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TICOM/D-51

TRANSLATIONS OF MISCELLANEOUS DOCUMENTS FROM PERS. Z.S. ARCHIVES

Attached are translations made at A.S.A. WASHINGTON of:

(a) Seven miscellaneous documents of general cryptanalytic interest.

(b) A report by Professor Dr. H. ROHRBACH on Sequences in Permutations

from the captured archives of Section Pers. Z.S. of the German Foreign Office at BURGSCHEIDUNGEN.

(TICOM Document Nos.380 and 1282).

TICOM

No. of Pages: (a) 8
(b) 9

30th October, 1945

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Mr. Foss
S.A.C. for Captain Filby

~~T O P S E C R E T~~

Items from Files of Foreign Office - Cryptographic Section

		Page
1.	Chronological Sequence of German Code Book.	1.
2.	Inquiry regarding a Machine (probably intended to produce the following Schablonen?)	2.
3.	Request for funds and paper to produce Forms for a Secret Cipher System.	2.
4.	The Tangenstafel (Additive book).	3.
5.	'Charges Saver 1926'	3.
6.	Numbering devices, presses, etc. (For producing one time pads?)	4.
7.	Various items of equipment considered or owned by the Code Section.	8.

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TICOM 1282

GERMAN CODE BOOKS

Berlin 22 IV 1925
The Kuerzungssatzbuch (condensing code), now about three years old, is out of date and used up. The manuscript of this new code book, which is a real improvement over the old one in respect to potential saving in telegraph charges and in practical use, is now complete. It has been shown to E.S. Mittler and Son, which firm printed the old book, for an estimate which is enclosed.

It is respectfully requested that authorization for printing by E.S. Mittler and Son be given as promptly as possible.

(signed) Selchow.

(Sheet 294)

To Section I B.

[This Code Nr. 2 was printed and put into use as shown by the next item]

The German Code Book (Nr. 2) which was printed in an edition of 500 copies and distributed to offices abroad 1 Sept. 1925 for immediate use is now used up to the point where only 50 copies remain in stock.

At the normal rate of use these will be exhausted within a year.

I request making provision so that when the present stock is used up no break in supply of codes be permitted to occur.

Berlin 11 Oct.1929
(signed) Mattelot (?) (Sheet 301)

German Code Book Nr. 3 and a new English Code Book (Nr.1) was requested 15 Nov.1930 and estimates submitted. A supplement was ordered (Ergaenzungsheft Nr.1) and bill was rendered 24 Sept.1931:

700 DeutschesSatzbuch 3 52 1/2 sheets 8vo
300 extra of 1st 16 pp
50 extra last 22 pp
500 Ergaenzungsheft 1 total bill 13595 RM. (Sheet 128)

The English Code Book was put out at about this time in 100 copies 23 1/2 sheets 8vo at a price of about 3030 RM. (Cf.Sheet 131)

By Nov.1930 the French Code Book had been reprinted once, dates of original printing and second printing not found. (Original 1927).

(Deutsches Satzbuch Nr.4 was issued with cover letter dated 28 Aug. 1940 and used up to the surrender in 1945).

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13 IX 45
R.W.P.

Note: When the "ten letter " codeword went out a new 8 page set of instructions for Code Book Nr.3 was issued on or about 14 July 1933.

(Sheets 139-140)

Instructions for English Code (Sheets 3-10) and German Code-book Nr. 3 (Sheets 11-26).

Ticom 1232 Berlin 8 Feb. 1926.

Referring to my conversation today with Lr. V.D.W. Krause I request remittance of 600,00 R.M. for production of Schablonen (Forms) for a secret cipher system.

I request that the necessary 5000 sheets of Konzeptpapier (low grade or scratch paper) be placed at my disposal through the Procurement Office.

(Signed) Selchow.

Ticom 1282 Berlin 13 Jan. 1925

Referring to your conversation with R. R. Langlotz I beg to inform you that for some special work (copying down cipher telegrams) (the procurement) of a Mercedes Elektron - with electric ..... operation and Zeilenschaltung (line spacing ??) is under consideration. The machine would have to have

1 row of numbers 0 - 19
spacing set for groups of five (?)

Between each two numbers 1 space, after each 5th number a double space. The complete form in which cipher telegrams are written down is shown by the enclosure.

I should appreciate a prompt reply whether your firm is in a position to deliver this special machine and, if so, how much it would cost.

To Walter Neumann
Representative of the Firm
Ferdinand S .....

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~~T O P S E C R E T~~

TICOM 1282 Tangenstafel (Sheets 94-111)

The additive book, which was first called SINUS (or SINUS) Tafel but changed to TANGENSTAFEL before printing, was issued in an edition of 250 copies, the bill for 5015 RM. is dated Nov. 6 1936.

Proof of 0000-0049 (105-106) and sample pages 57-58 - lines 2800-2899 (110-111). Printed at the Reichsdruckerei.

A conversion table (Umsatztafel) to be pasted inside the cover of each copy is mentioned (101).

(The GRADTAFEL, a similar book of additive, was first noted in traffic 10 IV 42, long after the records included in this document).

13 IX 45
R.W.P.

TICOM 1282

Charges Saver (Gebuehrensparer) 1926

To save on message charges under the "ten-letter" code rule conversion tables were issued. These were in two parts:

a three digit - three letter table for 000 - 999 and
a two digit - two letter table for 00 - 99

so arranged as to convert number groups into "pronounceable" words of five letters which were then combined in pairs to make the text of the message as transmitted.

There is also mention of an "Umsatztafel" (conversion table) but no instructions as to use were noted.

The Gebuehrensparer 1926 went out of use when the "ten letter" codeword went out, or it may have been supplanted by a later edition which went out in 1933.

(Sheets 139-140)

(Sheets 33-51)

13 IX 45
R.W.P.

~~T O P S E C R E T~~

~~T O P S E C R E T~~

Re TICOM 1282

GEE Machine. - Pad printer (?)

A British source states that German Government in 1925 purchased from Loranco Ltd. Engineers a cipher device for printing numerical (additive) keys and that a second was purchased in 1928 and a third in 1932. The records included in Ticom 1282 made no mention of British machines.

Machines which apparently were for this purpose were purchased from the German firms Otto Krebs and Clemens Mueller in 1925, 1927 and 1933, these were for the Cipher Section of the Foreign Office.

Apparently high speed job presses (Tiegeldruck-Schnell-presse) with automatic (pneumatic) feed - aside perhaps from the first - were fitted with special frames holding 240 steel wheels with the digits engraved thereon according to specifications (in 1927 an estimate was given for sets of 250 wheels - purpose not evident).

The chief items dealing with purchases and repairs may be summarized chronologically:

1925	Numerierwerk delivered 16 Oct.			
	Offer of one with 48 Werke (groups of five wheels?)	14 Nov.	5000 RM.	
	Recommendation that press built by Maschinenwerk Augsburg-Nuernberg (Akzidenzpresse) be purchased.			
1926	Offer of new press (few details)	24 Nov.		
	Estimate: motor, belt, etc.	29 Nov.	1142.50	RM.
1927	estimate on frame and 2 sets of wheels (480)			
	bill (for same?)	22 Oct.	17800	RM
	Offer of (frame?) 250 wheels - 3 types -	19 May		
1928	12 June: Report that it is impossible to get out cipher material promptly with present equipment			
1929	estimate on 240 new wheels	19 Jan.		
1930	paper moisteners and motor	1 Sept.	267.50	RM
1931	repair bill	20 Jan.	483.75	RM
	repair bill, includes moving machine to another room,	22 May	6275.00	RM
1933	repairs on 1927 machine		302.75	RM
	offer 10 Jan. of special frame and wheels supplied by June		11200.00	RM
1935	repair bill	15 March	833.60	RM
	motor &	16 May	663.00	RM
	repairs	13 June	310.50	RM
	repairs	18 Oct.	436.00	RM
	overhaul needed	Dec.		
1936	repairs on 1933 machine	Dec	36.00	RM

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TICOM 1282 Selected Items

Estimate for a Numerierwerk 19 May 1927

Offer 1:
1. A Numerierwerk consisting of 250 cipher wheels engraved to specifications.
2. Shaft bearings divided
3. each row of numbers interchangeable
4. each group of numbers interchangeable
5. frame so constructed that each row of numbers can be removed
6. 5 spare cipher wheels engraved to specifications
7. /ugschiene (drew-bars??) to run on ball bearings
(written in: Rahmenzug doppelseitig)

Offer 2:
1. A Numerierwerk consisting of 250 cipher wheels engraved to specifications,
2. individual wheels interchangeable
3. each row of numbers interchangeable
4. each group of numbers interchangeable
5. divided shaft bearing and divided shaft
6. Rahmenzug doppelseitig, ending in middle, Zugschiene to run on ball bearings
7. fine adjustment of each row of numbers by micrometer screw
8. corresponding change in the Zug of the machine
9. set of spare cipher sheels with Vorder-und Hintergreifer, 6 spare shafts.

Offer 3:
1. A Numerierwerk consisting of 250 cipher wheels engraved to specifications.
2. individual wheels interchangeable
3. 250 additional cipher wheels engraved to specifications with Vorder - und Hintergreifer to go with them.
4. wheels mentioned under 3 likewise to be interchangeable individually.
5. divided shaft bearings and shaft
6. each individual group of numbers interchangeable
7. automatic Verstellung (stopping ?) for one of each group of 5 wheels.
8. device for Verstellung (stopping) the carrying (Transport) of each number at a different count.
9. each row of numbers interchangeable
10. Rahmenzug doppelseitig, ending in middle, Zugschiene to run on ball bearings.
11. fine adjustment of each row of numbers by micrometer screw
12. each row of numbers can be removed from frame separately
13. corresponding alteration of Zug of the machine
14. various spare parts

(Submitted by Clemens Mueller, no price given on sheets)
(Sheets 161-164)

Picture of Tiegeldruck-Schnellpresse Original "Brillant"
(Sheet 160)

Excerpt of bill (seemingly for 136331.00 RM) asking balance
(Sheet 165)

(No information as to what was purchased)
1 June 1931

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~~T O P S E C R E T~~

TICOM 1282

Pad printer (?)

In the course of time some parts of Numeriewerk Nr.1 have gotten so they no longer function properly. Furthermore all the Vorder- und Hintergreifer (front and rear pawls or fingers?) need to be replaced. Moreover the frame for the matrices (printing wheels) is slightly bent and must be straightened and the play of the individual Werke (sets of five wheels?) needs adjusting. Several printing wheels need to be reengraved.

These repairs are needed because about 1000000 prints have been made. (Sheet 120)

In consequence of very heavy use some parts of the Secret Printing machine (Geheim-Druckmaschine), in particular the Numerierwerk, are no longer in dependable operating condition and must therefore be repaired. According to oral statements of the firm C.Mueller the cost of going over the defective parts will be some 450.00-500.00 Rm.

Submitted to Ministerial Director Koester with the request that this expenditure for repairs be approved.

Berlin 20 Jan. 1932.
(signed) Selchow. (Sheet 119)

10 Nov. 1932.

Referring to drawings and calculations made, I offer the following estimate:

1. Special frame for high speed press with 240 automatically steppable steel wheels, each wheel rotatable according to specifications and all interchangeable (added: with Vordergreifer which are all interchangeable and differently abgestuft 11400 RM
2. all wheels eleven part but with Vordergreifer which are all interchangeable and differently abgestuft 13500 RM
3. all wheels twelve part but with (Vordergreifer which are all interchangeable and differently abgestuft and equipped with cranks) 15400 RM

I guarantee perfect operation and request order

Respectfully

Otto Krebs (Sheet 66)

Berlin 3 April 1933

Construction of the special framework for speed press ordered from Otto Krebs on 27th and confirmed on 31st March 1933:

Special framework for special press with 240 ten part, automatically steppable steel wheels, drive double Zug ("double numbering"?), engraved and rotable according to specifications of Foreign Office, all interchangeable; with Vordergreifer which are interchangeable and abgestuft according to specifications of the Foreign Office and are equipped with cranks; also delivery of 12 spare Vordergreifer of like kind.

~~T O P S E C R E T~~

The above 240 wheels are being made interchangeable with those now in the Foreign Office
For the firm of Otto Krebs. (Sheet 46)

Secret traffic, which had previously been relatively light, has increased greatly in the past month or so and in view of the Military Attache material and of the present political situation may be expected to become very heavy.

Even with light traffic it would have been necessary so I pointed out in my estimate of 3 Dec. 1932, to procure another modern special numbering machine to insure speedy production and security of keys.

The using up of keys, accelerated by the increased traffic makes this necessary now if a prompt supply of keys is to be assured because with present resources no reserve can be built up and in case of any break down an interruption of supply would ensue.

Therefore I earnestly request a grant of RM 11200 for the purchase of a machine according to the attached estimate.

Submitted to Referat I D.

Berlin 27 March 1933.
(Signed) Slechow. (Sheet 63)

Estimate: Repair of press

Berlin 4 Dec. 1935

At the request of the cipher section the following estimate on repairs to the Schnellpresse-Spezialtiegel delivered by us in December 1926 is submitted:

Walzenstuhl-Gegendruckfedern* and appurtenant parts to be replaced
Ausrueckung* des Walzenstuhls to be adjusted
Zughaken und Gestaenge* to be fitted
1 pair narrow and 1 pair wide Greifer*, including screws and Huelsen*, to be made, compressed air tubes and connections for the feed to be replaced along with new suction cups,
device to be constructed for setting feed arm so that when abgefahren it cannot change position,
valve on air pump to be repaired and adjusted.

The foregoing, including material and mounting and parts enumerated, would come to RM 227.00.

Mit deutschem Gruss
Clemens Mueller (Sheets 77-78)

* Walzenstuhl, probably the mounting of the ink rolls
Gegendruckfedern - counter springs
* Ausrueckung - perhaps release of inking rolls when press runs empty
* Zughaken und Gestaenge - ?? draw hook and bars ??
* Greifer - fingers, pawls? Huelsen - may be collars here
* abgefahren - ?? abfahren - to drive off, might be : gotten off the track ??

~~TOP SECRET~~

Various items of equipment considered or owned by the Code section of the Foreign Office.

ROTAPRINT: used to multilithograph incoming messages (159) in need of repair: photo (325): shield needed to protect female operator (102-104): repairs necessary on both "old and new" machines - 1936 (81-95): (see also 52-53, 339).

Electric Gluepot: (126-128, 302-308) cost 30-35 RM.

Perforators: Several styles considered (53, 129-134, 139-141, 160). Pictures, and one plate of types of perforation.

Cutters (trimmers): (47, 139-141).

Paper shredder: several types considered, some used. (176, 249: 264, 330-333, 335: 126-128).

Envelope folder: (54-57, 66-69).

Leica Film Viewer, 72 RM.

AKZIDENZKASTEN (job printers table with compartments for frames at one end and type fonts??) also some type. (54-57, 66-69).

Dust remover (pneumatic ?) for type equipment, needed to decrease danger of lead poisoning. (311)

Dailygraph (?): seemingly a telephone recorder, price 1330 RM. was considered for Berlin-Berne circuit but apparently not purchased.

13 IX 45
R.W.F.

~~TOP SECRET~~

~~T O P S E C R E T~~

15-7-44.
TICOM 380

On Sequences in Permutations

by Prof. Dr. H. Rohrbach

1. Introduction

Dr. Dueball published in the issue of 26.5.42 of these 'Papers" some results on permutations, in particular formulae on the number $P_k(n)$ of permutations of n elements with k fixed elements. The determination of $P_k(n)$ is easily carried back to that of $P_o(n)$ of the permutation without fixed elements.

In what follows I should like to give a counterpart, namely to derive the number $S_k(n)$ of permutations of n elements with k sequences, determination of which will be carried back to that of $S_o(n)$ of the permutations without sequences. In this connection I say the permutation

$$P = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}, \quad n \geq 2,$$

contains one sequence if for any i $(1 \leq i \leq n-1)$ the relation

(1) $a_{i+1} = a_i + 1$ holds, E.g., the permutations

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 5 & 6 & 1 \end{pmatrix}$$

each have two sequences.

The formulae result:

$$\left.\begin{aligned} (2)\quad S_k(n) &= \binom{n-1}{k} S_o(n-k), \\ (3)\quad S_o(n) &= \sum_{i=o}^{n-1} (-1)^i \binom{n-1}{i} (n-i)! \end{aligned}\right\} \quad (n \geq 2).$$

~~T O P S E C R E T~~

Clearly $0 \leq k \leq n-1$; for $k = n-1$ in (2) $S_0(1) = 1$ is to be inserted. For every $n \geq 2$ $S_{n-1}(n) = 1$, since the identity permutation is the only one with n-1 sequences.

Furthermore if $E_k(n)$ means the relative frequency of the permutations of n elements with k sequences, then as application is indicated:

$$(4) \quad \lim_{n \to \infty} E_k(n) = \frac{1}{k!} \frac{1}{e}$$

in particular

$$(5) \quad \lim_{n \to \infty} E_0(n) = \frac{1}{e} = 0,3678 \ldots$$

For large values of n, therefore, approximately one third of all permutations are free from sequences, i.e., at no point show even two elements standing in their natural sequence.

This result should be of interest perhaps for theoretical studies of ciphers. This is the reason I give it here. The inspiration to occupy myself with the question of sequences in permutations came from Eduard Baron von Stackelberg, a fatherly friend, who had found the solution empirically but unfortunately did not live to see it worked out mathematically. He died in 1942 as the result of an accident. I wish to thank my associate in Prag, Dr. Paul Armsen, for his substantial aid in deriving the proofs.

2. Reduction of $S_k(n)$ to $S_0(n)$

1. Introduction of the basic permutation

In permutation P let us designate the elements $a_1, a_2, \ldots, a_n$ as new, according to the following determination:

Let (6)

$$a_1 = b_{10},\ a_i = b_{jk},\ a_{i+1} = \begin{cases} b_{j,k+1}, & \text{when } a_{i+1} = a_i + 1 \\ b_{j+1,0}, & \text{when } a_{i+1} \neq a_i + 1 \end{cases}$$

Then the second line of P reads:

(7) $b_{10}, b_{11}, \ldots, b_{1k_1}, b_{20}, b_{21}, \ldots, b_{2k_2}, \ldots, b_{m0}, b_{m1}, \ldots, b_{mk_m}$.

In this are the k_i whole numbers ≥ 0 by which the line (7) is broken up into m parts with lengths $k_i + 1$ $(i = 1, 2, \ldots, m)$. If $k_i > 0$, then the i^{th} part consists of k_i sequences; if $k_i = 0$ then the i^{th} part consists solely of the number b_{i0}, which forms no sequence with the adjacent number $b_{i+1,0}$ according to (6). All told the line (7) contains

$$k = \sum_{i=1}^{m} k_i$$

sequences and $m = n - k$.

Let us remove from (7) the elements $b_{10}, b_{20}, \ldots, b_{m0}$, take $b_{i0} = b_i$, number the b_i's according to magnitude and replace each b_i by the number c_i thereby determined. Then the statement 1 holds: The permutation

$$Q = \begin{pmatrix} 1 & 2 & \ldots & m \\ c_1 & c_2 & \ldots & c_m \end{pmatrix}$$

of elements 1, 2, ..., m contains no sequences.

Proof. $b_{ik_i} = b_{i0} + k_i = b_i + k_i$ hence from (6) it follows that

(8) $b_{i+1} \neq b_i + k_i + 1$.

On the other hand by construction the numbers $b_i + k_i + 1$ $(i = 1, 2, \ldots, m)$ yield, when regarded as modulo n, the numbers

b_i ($i = 1,2,\ldots,m$) and indeed $b_i + k_i + 1$ is the least b_j greater than b_i. For the transition from the b_i's to their numbers c_i, therefore, it holds true that $b_i + k_i + 1$ has the number $c_i + 1$. Due to (8) then

$$c_{i-1} \neq c_i + 1 \quad \text{for } i = 1,2,\ldots,m,$$

i.e., the permutation Q is free from sequences (cf.(1)).

This permutation Q determined uniquely by P may be called the basic permutation /Stammpermutation/ to P. Conversely to a basic permutation Q of m elements there may belong several permutations P of $n = m$ elements, and since n has no upper limit for every fixed Q there is an infinite number of permutations which have Q as basic permutation.

2. Reduction and dilation

If in P we strike out any one of the numbers b_{ij} with $j \neq 0$ and replace all $b_{rs} > b_{ij}$ by $b_{rs} - 1$ we get a new permutation P' of n-1 elements. If Q and Q' are the basic permutations to P and P' respectively, then

(9) $$Q = Q'.$$

Comparing line (7) with the corresponding line of P'(designating its elements by a prime mark) we see that

$$b'_i = b_i \quad \text{for } b_i < b_{ij},$$

$$b'_i = b_i - 1 \quad \text{for } b_i > b_{ij}.$$

The case $b_i = b_{ij}$ is not possible since $j \neq 0$. Hence the sequence and order of magnitudes of b_i are preserved; consequently Q' has the same elements in the same sequence as Q. Hence (9) is proven.

In precisely the same manner it is shown that if one introduces into P after some element b_{ij} (ij being any values) the number $b_{ij} + 1$ as an element and replaces all $b_{rs} \geq b_{ij} + 1$ (except the one just introduced) by $b_{rs} + 1$, the new permutation P^+ of n+1 elements belongs to Q, the basic permutation of P.

Of the two operations described let us term the transition $P \longrightarrow P'$ a reduction and the transition $P \longrightarrow P^+$ a dilation. Then statement 2 holds:

To every reduction there can be uniquely associated a dilation in such fashion that the two operations cancel one another.

This is a trivial matter. For if a reduction $P \longrightarrow P'$ is given, one may choose as dilation $P' \longrightarrow (P')^+$ that one which restores to its original position the element stricken out in the reduction and vice versa. Statement 2 is only worth noting because with the given P and P' there may be several reductions which transform P into P' (if, for example, in breaking up (7) parts with a length of $k_i > 2$ occur), and P can also pass into a given P^+ through several dilations.

3. Setting up the formula

Now Statement 3 holds good. If $S_k(n)$ is the number of permutations of n elements with k sequences, $(n \geq 2)$, then

$$S_k(n) = \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} S_0(n-k). \tag{10}$$

Proof. With the permutation P with k sequences k reductions are possible, with each P' there are k - 1,..., with each $P^{(k-1)}$ there is still one. By k successive reductions

P passes into its basic permutation Q and this transition can occur in k! ways. Conversely with each permutation Q of m elements m dilations are possible, with each $Q^{\dagger}$ there are m + 1, etc. From a given Q with m elements there will always arise by k successive dilations a permutation of m + k = n elements with k sequences having Q as basic permutation and this transition can occur in $m(m-1)\ldots(n-1)$ ways. The number of all permutations Q of m elements which are free from sequences is now by definition equal to $S_o(m)$. Every such Q with m = n - k is, according to statement 1 possible as basic permutation to P, hence by statement 2 it follows that

$$k!\, S_k(n) = m(m-1)\ldots(n-1)\, S_o(m),$$

from which one gets the formula of statement 3 by writing m = n - k.

3. Determining $S_o(n)$

If we divide the permutations of n + 1 elements into classes according to the number of sequences they contain, then

$$(n+1)! = \sum_{k=o}^{n} S_k(n+1)$$

and from this by (10)

$$(11)\quad (n+1)! = \sum_{k=o}^{n} \left\{ {n \atop k} \right\} S_o(n+1-k) = \sum_{i=o}^{n} \left\{ {n \atop i} \right\} S_o(i+1).$$

I now prove

Statement 4. For n = 2

$$(12)\quad S_o(n) = \sum_{i=o}^{n-1} (-1)^i \left\{ {n-1 \atop i} \right\} (n-i)!$$

Proof. Formula (12) is correct for n = 2. For then both sides have the value 1. Hence one can accept this as already proven. From (11) it follows by splitting off the summand for i = n at the right,

$$S_o(n+1) = (n+1)! - \sum_{i=o}^{n-1} \binom{n}{i} S_o(i+1)$$

and from this, using the deduced assumption (12),

$$(13) \quad S_o(n+1) = (n+1)! - \sum_{i=o}^{n-1} \binom{n}{i} \sum_{j=o}^{i} (-1)^j \binom{i}{j} (i+1-j)!$$

If one conceives (12) as written for n + 1 instead of n and then compares it with (13), one sees that for proof of (12) by the deduction from n to n + 1 it is enough to show the following relation:

$$(14) \quad \sum_{i=o}^{n-1} \binom{n}{i} \sum_{j=o}^{i} (-1)^j \binom{i}{j} (i+1-j)! = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (n+1-i)!$$

For brevity let us designate the left side of (14) as L(n) and the right side as R(n). If we take i = r-1 and j = r-s, it follows that

$$L(n) = \sum_{r=1}^{n} \binom{N}{r-1} \sum_{s=1}^{r} (-1)^{r-s} \binom{r-1}{s-1} s!$$

Since for r<s the summands of the inner sum disappear, the summation via s may be extended as far as one wishes (say to n) and the sequence of summations substituted. Then

$$L(n) = \sum_{s=1}^{n} (-1)^s s! \sum_{r=1}^{n} (-1)^r \binom{r-1}{s-1} \binom{n}{r-1}.$$

Moreover it follows for n+1-i = s that

$$R(n) = \sum_{s=1}^{n} (-1)^{n-s} \binom{n}{s-1} s!$$

From the two there results

$$R(n) - L(n) = \sum_{s=1}^{n} (-1)^s\, s! \left[(-1)^n \left\{ \begin{matrix} n \\ s-1 \end{matrix} \right\} - \sum_{r=1}^{n} (-1)^r \left\{ \begin{matrix} r-1 \\ s-1 \end{matrix} \right\} \left\{ \begin{matrix} n \\ r-1 \end{matrix} \right\} \right]$$

$$= \sum_{s=1}^{n} (-1)^s\, s! \sum_{r=1}^{n+1} (-1)^{r+1} \left\{ \begin{matrix} r-1 \\ s-1 \end{matrix} \right\} \left\{ \begin{matrix} n \\ r-1 \end{matrix} \right\} .$$

If for brevity we designate the inner sum here as $M(s)$, it only remains to be shown that $M(s)$ for $s = 1,2,\ldots,n$ disappears. For this take $s-1 = i$ and $r-1 = j$ in $M(s)$. It then follows that

$$M(s) = \sum_{j=0}^{n} (-1)^j \left\{ \begin{matrix} j \\ i \end{matrix} \right\} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} = \frac{1}{i!} \sum_{j=i}^{n} (-1)^j \left\{ \begin{matrix} n \\ j \end{matrix} \right\} j(j-1)\ldots(j-i+1)$$

$$= \frac{1}{i!} \left[\frac{d^i (1-x)^n}{dx^i} \right]_{x=1} = 0.$$

This holds for $i = 0,1,\ldots,n-1$, hence $s = 1,2,\ldots,n$. Thus statement 4 is proven inductively.

4. Relative frequency of permutations with k sequences

Among the $n!$ permutations of n elements there are $S_k(n)$ with k sequences. For the relative frequency $E_k(n)$ of the permutations with k sequences it is therefore true that

$$E_k(n) = \frac{S_k(n)}{n!} .$$

From (10) and (12) it results that

$$E_k(n) = \frac{1}{n!} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} \sum_{i=0}^{n-k-1} (-1)^i \binom{n-k-1}{i} (n-k-i)\,!$$

$$= \sum_{i=0}^{n-k-1} (-1)^i \frac{p=k+i}{n.k!i!}$$

$$= \frac{1}{k!}\sum_{i=0}^{n-k-1}(-1)^i\frac{1}{i!} - \frac{1}{k(k-1)!}\sum_{i=0}^{n-k-1}(-1)^i\frac{1}{i!}$$

$$+ \frac{1}{n.k!}\sum_{i=1}^{n-k-1}(-1)^{i-1}\frac{k}{(i-1)!}$$

If we let n approach ∞, the three sums occurring here are limited, since each of them for n approaching ∞ approaches the limit value e^{-1}. Thus

$$(15) \qquad \lim_{n \to \infty} E_k(n) = \frac{1}{k!}\,\frac{1}{e} .$$

Taking the left side of (15) equal to W(k), it results that

$$\sum_{k=0}^{\infty} W(k) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!} = \frac{1}{e}\,e = 1 ,$$

as must be the case.

With (10), (12) and (15) the three assured formulae (2), (3) and (4) of the introduction are proven and hence also the deduction (5).

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Translated by Dr. Ray W. Pettengill, 4 August 1945.